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Problem: For positive integers m, n and k with $k \geq 2$,

Prove

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (\min(i_1, \cdots, i_k))^m = \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i - n^i\right) \sum_{j=1}^{n} j^{k+m-i}$$

Solution:

Let us denote $[n]_j := \{j, j+1, \cdots, n\}$ for each $j \in \{1, 2, \cdots, n\} := [n]$ and $[n]_{n+1} = \phi$. Define $[n]_j^k := \{(i_1, \cdots, i_k) | j \leq i_1, \cdots, i_k \leq n\}$.

Also, let us denote the identity function $\mathbf{1}_{[j,k]} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$

Hence,

$$
\begin{aligned}
\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}(\min(i_1,\cdots,i_k))^m &= \sum_{(i_1,\cdots,i_k)\in[n]^k}(\min(i_1,\cdots,i_k))^m \\
&= \sum_{j=1}^{n}\sum_{(i_1,\cdots,i_k)\in[n]^k} j^m \times \mathbf{1}_{[j,\min(i_1,\cdots,i_k)]} \\
&= \sum_{j=1}^{n} j^m \sum_{(i_1,\cdots,i_k)\in[n]_j^k\setminus[n]_{j+1}^k} 1 \\
&= \sum_{j=1}^{n} j^m \times \operatorname{card}\left|[n]_j^k \setminus [n]_{j+1}^k\right| \\
&= \sum_{j=1}^{n} j^m \left((n+1-j)^k-(n-j)^k\right) \quad (\text{reindex: } j \mapsto n+1-j) \\
&= \sum_{j=1}^{n}(n+1-j)^m \left(j^k-(j-1)^k\right) \\
&= \sum_{j=1}^{n}((n+1-j)^m-(n-j)^m)j^k \\
&= \sum_{j=1}^{n}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}((n+1)^i-n^i)j^{k+m-i} \\
&= \sum_{i=1}^{m}(-1)^{m-i}\binom{m}{i}((n+1)^i-n^i)\sum_{j=1}^{n} j^{k+m-i}
\end{aligned}
$$

<u>N.B.:</u>

$$
\begin{aligned}
\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}(\min(i_1,\cdots,i_k))^m &= \sum_{j=1}^{n} j^m \left((n+1-j)^k-(n-j)^k\right) \\
&= \sum_{j=1}^{n} j^k \left((n+1-j)^m-(n-j)^m\right) \\
&= \sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}(\min(i_1,\cdots,i_m))^k
\end{aligned}
$$